Step 2 Relate $\psi(x)$ to $\zeta'(s)/\zeta(s)$.

For $x > 0$ define

$$E(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ 0 & \text{if } x < 1. \end{cases}$$

Then

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{n=1}^{\infty} \Lambda(n) E\left(\frac{x}{n}\right). \tag{16}$$

Theorem 6.17 *If $x > 0$ and $c > 0$ then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)} ds = \left(1 - \frac{1}{x}\right) E(x). \tag{17}$$

Proof Let $\mathcal{C} = \mathcal{C}(0, R)$ be the circle centre the origin, radius $R > \max(1, c)$.

Let the vertical line $\operatorname{Re} s = c$ meet the circle at points $c \pm it_R$.

Let $\mathcal{L}_R$ be the vertical line segment from $c - it_R$ to $c + it_R$ and let $\mathcal{C}_1$, $\mathcal{C}_2$ be the sections of the circle $\mathcal{C}$ lying to the left and right of this line, respectively. so

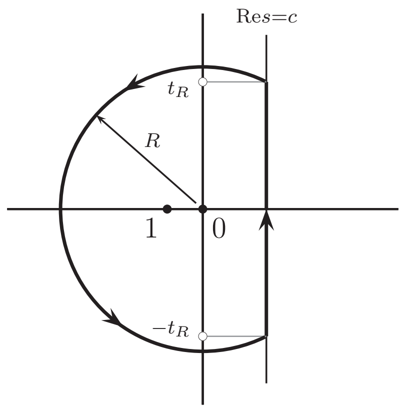
$$\mathcal{C}_1 = \{s \in \mathcal{C} : \operatorname{Re} s \leq c\} \quad \text{and} \quad \mathcal{C}_2 = \{s \in \mathcal{C} : \operatorname{Re} s \geq c\}.$$

For any regular path $\Gamma \subseteq \mathbb{C}$ write

$$I(\Gamma) = \frac{1}{2\pi i} \int_{\Gamma} \frac{x^s}{s(s+1)} ds,$$

so, in particular, the left hand side of (17) is $\lim_{R\to\infty} I(\mathcal{L}_R)$.

Assume $x \geq 1$ and consider $I(\mathcal{L}_R \cup \mathcal{C}_1)$.



Partial fractions show that

$$\frac{1}{s\,(s+1)} = \frac{1}{s} - \frac{1}{s+1} \tag{18}$$

so the integrand of $I(\mathcal{L}_R \cup \mathcal{C}_1)$ has two simple poles, at $s = 0$ and -1 which lie *inside* $\mathcal{L}_R \cup \mathcal{C}_1$ since $R > 1$. The residues at the poles are

$$\begin{aligned} \operatorname*{Res}_{s=0} \frac{x^s}{s\,(s+1)} &= \lim_{s\to 0}\,(s-0)\,\frac{x^s}{s\,(s+1)} = 1, \\ \operatorname*{Res}_{s=-1} \frac{x^s}{s\,(s+1)} &= \lim_{s\to -1}\,(s+1)\,\frac{x^s}{s\,(s+1)} = -\frac{1}{x}. \end{aligned}$$

So, by Cauchy's Theorem, (stated as the integral around the boundary of a finite region equals the sum of the residues of poles within the region)

$$1 - \frac{1}{x} = I(\mathcal{L}_R \cup \mathcal{C}_1) = I(\mathcal{L}_R) + I(\mathcal{C}_1) \tag{19}$$

for all $R > 1$. We will be letting $R \to \infty$.

On the circle $\mathcal{C}_1$ we have $|s| = R$ and

$$|s+1| \geq |s| - 1 = R-1,$$

having used the triangle inequality. Also, on $\mathcal{C}_1$ we have $\operatorname{Re} s \leq c$ and thus, since we are assuming $x \geq 1$ we have $|x^s| = x^\sigma \leq x^c$. Hence

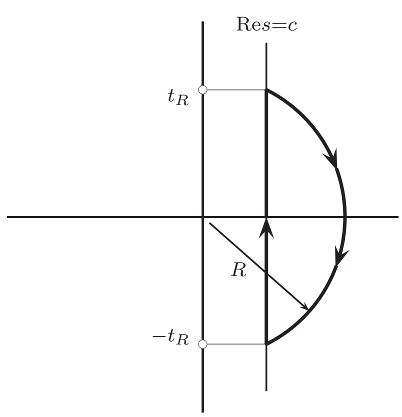
$$\begin{aligned} |I(\mathcal{C}_1)| &= \left|\frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{x^s}{s\,(s+1)}ds\right| \leq \frac{1}{2\pi}\int_{\mathcal{C}_1} \frac{x^c}{R\,(R-1)}\,|ds| \\ &\leq \frac{1}{2\pi}\frac{x^c}{R\,(R-1)}2\pi R \\ &= \frac{x^c}{R-1}, \end{aligned}$$

which tends to zero as $R \to \infty$. Hence letting $R \to \infty$ gives

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s\,(s+1)}ds &= \lim_{R\to\infty} I(\mathcal{L}_R) \\ &= \lim_{R\to\infty} I(\mathcal{L}_R \cup \mathcal{C}_1) \\ &\qquad \text{by (19) and } \lim_{R\to\infty} I(\mathcal{C}_1) = 0 \\ &= 1 - \frac{1}{x} = \left(1 - \frac{1}{x}\right) E\,(x) \end{aligned}$$

since $x \geq 1$.

Assume next that $0 < x < 1$ and consider $I(\mathcal{L}_R \cup \mathcal{C}_2)$.



The integrand (18) has **no** poles inside this contour and so, by Cauchy's Theorem, $I(\mathcal{L}_R \cup \mathcal{C}_2) = 0$ for all R.

On $\mathcal{C}_2$ we have $\operatorname{Re} s \geq c$ but now $x < 1$ so $|x^s| = x^\sigma \leq x^c$ again. Thus we recover the same bound $|I(\mathcal{C}_2)| \leq x^c/(R-1)$ which tends to 0 as $R \to \infty$.

Hence, if $x < 1$, then

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)}ds &= \lim_{R\to\infty} I(\mathcal{L}_R) \\
&= \lim_{R\to\infty} I(\mathcal{L}_R \cup \mathcal{C}_2) \\
&\qquad \text{by (19) and } \lim_{R\to\infty} I(\mathcal{C}_2) = 0 \\
&= 0 = \left(1 - \frac{1}{x}\right) E(x)
\end{aligned}$$

since $0 < x < 1$. ■

Question about the proof. Why do we choose the contour $\mathcal{L}_R \cup \mathcal{C}_1$ when $x \geq 1$ and $\mathcal{L}_R \cup \mathcal{C}_2$ when $x < 1$?

Answer In the proof we made use of $x^\sigma \leq x^c$.

If $x \geq 1$ then, to get an upper bound on x^σ, we need an *upper bound* σ, thus we keep s to the *left* of the line $\operatorname{Re} s = c$, i.e. $s \in \mathcal{C}_1$.

If $x < 1$ then to get an upper bound on x^σ we need a *lower bound* on σ, and so we keep s to the *right* of the line $\operatorname{Re} s = c$, i.e. $s \in \mathcal{C}_2$.

Theorem 6.18 *Suppose that* $\sum_{n=1}^\infty a_n n^{-s}$ *is absolutely convergent for* $\operatorname{Re} s > 1$ *with sum* $D(s)$*. Let* $A(x) = \sum_{n\leq x} a_n$*. Then for* $c > 1$ *and* $x > 1$*,*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} D(s)\,\frac{x^{s+1}}{s(s+1)}ds = \int_1^x A(t)\,dt.$$

(Hand waving For All) **If** we could justify the interchanges of infinite integrals with infinite sums **then** we could say

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{a_n}{n^s}\frac{x^{s+1}}{s(s+1)}ds &= x\sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)}ds \\
&= x\sum_{n=1}^{\infty} a_n \left(1 - \frac{n}{x}\right) E\left(\frac{x}{n}\right) \\
&= \sum_{n\leq x} a_n (x-n) = \int_1^x A(t)\,dt.
\end{aligned}$$

Proof Full details in the appendix. ■

A naive application of Theorem 6.18 would be with $D(s) = \zeta'(s)/\zeta(s)$, but, as we will see below, this has a pole at $s = 1$ which we would rather was not there.

Recall that if a function F is holomorphic in a region except for either a pole or zero at α then it can be written as

$$F(s) = g(s)(s-\alpha)^m,$$

with $g(s)$ holomorphic in a region containing α, and $g(\alpha) \neq 0$. Here $m \in \mathbb{Z}$ is the *order* of the singularity and is > 0 if α is a zero, and < 0 if α is a pole.

For $s \neq \alpha$, the derivative of F is, by the Product Rule,

$$F'(s) = mg(s)(s-\alpha)^{m-1} + g'(s)(s-\alpha)^m.$$

Then

$$\frac{F'(s)}{F(s)} = \frac{mg(s)(s-\alpha)^{m-1} + g'(s)(s-\alpha)^m}{g(s)(s-\alpha)^m} = \frac{m}{s-\alpha} + \frac{g'(s)}{g(s)}. \tag{20}$$

The term $g'(s)/g(s)$ is well-defined close to $s = \alpha$ since $g(\alpha) \neq 0$. The other term $m/(s-\alpha)$ has a simple pole (i.e. of order 1) at $s = \alpha$, with residue m.

We apply this with $F(s) = \zeta(s)$. From Theorem 6.12 we have

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{1+s}}du \qquad (21)\\ &= \frac{1}{s-1} + h(s) = \frac{g(s)}{s-1},\end{aligned}$$

where

$$h(s) = 1 - s\int_1^\infty \frac{\{u\}}{u^{1+s}}du \quad \text{and} \quad g(s) = 1 + (s-1)h(s).$$

In the notation above, $m = -1$ and thus (20) gives

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \frac{g'(s)}{g(s)}.$$

If this added to (21) we find that

$$\frac{\zeta'(s)}{\zeta(s)} + \zeta(s) = \left(-\frac{1}{s-1} + \frac{g'(s)}{g(s)}\right) + \left(\frac{1}{s-1} + h(s)\right) = \frac{g'(s)}{g(s)} + h(s)\,,$$

i.e., the poles cancel!

Write

$$F(s) = \frac{\zeta'(s)}{\zeta(s)} + \zeta(s) = \sum_{n=1}^{\infty} \frac{(-\Lambda(n)+1)}{n^s},$$

for $\mathrm{Re}\, s > 1$. It is to $F(s)$ that we apply Theorem 6.18, and in the notation of that result

$$A(x) = \sum_{n \le x} (-\Lambda(n)+1) = -\psi(x) + [x]\,.$$

Hence, for $c > 1$, Theorem 6.18 gives the **fundamental**

$$\int_1^x (\psi(t) - [t])\, dt = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, \frac{x^{s+1} ds}{s\,(s+1)}.$$

From a problem sheet you were asked to show that

$$\int_1^x [t]\, dt = \frac{1}{2}x^2 + O(x)\,.$$

Combine to get the **important**

$$\int_1^x \psi(t)\, dt = \frac{1}{2}x^2 - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, \frac{x^{s+1} ds}{s\,(s+1)} + O(x)\,.$$

Appendix for Step 2

We here replace a handwaving justification of Theorem 6.18 given in the lectures by its proof.

Theorem 6.18 *Suppose that* $\sum_{n=1}^{\infty} a_n n^{-s}$ *is absolutely convergent for* $\operatorname{Re} s > 1$ *with sum* $D(s)$*. Let* $A(x) = \sum_{n \le x} a_n$*. Then for* $c > 1$ *and* $x > 1$*,*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D(s) \, \frac{x^{s+1}}{s\,(s+1)} ds = \int_1^x A(t) \, dt. \tag{22}$$

Proof Write

$$x^s D(s) = G(s) + H(s) \, ,$$

where

$$G(s) = \sum_{n \le x} a_n \left(\frac{x}{n} \right)^s \quad \text{and} \quad H(s) = \sum_{n > x} a_n \left(\frac{x}{n} \right)^s .$$

Since $G\,(s)$ is only a *finite* sum we are justified in interchanging the summation and integration in

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s) \, \frac{x}{s\,(s+1)} ds &= x \sum_{n \le x} a_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s\,(s+1)} ds \\ &= x \sum_{n \le x} a_n \left(1 - \frac{n}{x} \right) \quad \text{by Theorem 6.17,} \\ &= \sum_{n \le x} a_n \, (x - n) = \int_1^x A\,(t) \, dt. \end{aligned}$$

Thus $G\,(s)$ has given the right hand side of (22), so the remaining $H(s) = x^s D(s) - G(s)$ will have to contribute nothing!

Consider now

$$\frac{1}{2\pi i} \int_{\mathcal{L}_R \cup \mathcal{C}_2} H(s) \, \frac{x}{s\,(s+1)} ds, \tag{23}$$

with notation from the proof of Theorem 6.17. Inside the contour $\mathcal{L}_R \cup \mathcal{C}_2$, the series $H(s)$ differs from $D(s)$ by only a finite number of terms and is thus absolutely convergent and has no poles. Therefore, by Cauchy's Theorem, the integral (23) is zero, i.e.

$$0 = \frac{1}{2\pi i} \int_{\mathcal{C}_2} H(s) \, \frac{x}{s\,(s+1)} ds + \frac{1}{2\pi i} \int_{\mathcal{L}_R} H(s) \, \frac{x}{s\,(s+1)} ds. \tag{24}$$

We have chosen $\mathcal{C}_2$ instead of $\mathcal{C}_1$ because for the terms $(x/n)^s$ seen in $H(s)$, we have $n > x$, i.e. $x/n < 1$ and so to get an *upper* bound on $(x/n)^\sigma$ we need a *lower* bound on σ. Thus we keep s to the *right* of the line $\operatorname{Re} s = c$, in other words, $s \in \mathcal{C}_2$. For such s we get

$$\left|\left(\frac{x}{n}\right)^s\right| = \left(\frac{x}{n}\right)^\sigma \leq \left(\frac{x}{n}\right)^c.$$

Justify the following use of the triangle inequality on an infinite sum by the fact that $\sum_{n=1}^\infty a_n n^{-s}$ converges absolutely. So

$$|H(s)| = \left|\sum_{n>x} a_n \left(\frac{x}{n}\right)^s\right| \leq \sum_{n>x} |a_n| \left(\frac{x}{n}\right)^\sigma \leq \sum_{n>x} |a_n| \left(\frac{x}{n}\right)^c$$

For $H(s)$ we have, by assumption, that $\sum_{n>x} |a_n|/n^c \leq \sum_{n=1}^\infty |a_n|/n^c$, which converges, to M say. Hence

$$|H(s)| \leq x^c \sum_{n>x} \frac{|a_n|}{n^c} \leq M x^c.$$

Using the arguments seen in the proof of Theorem 6.17,

$$\left|\frac{1}{2\pi i}\int_{\mathcal{C}_2} H(s)\frac{x}{s(s+1)}ds\right| \leq \frac{Mx^{c+1}}{R-1},$$

which tends to 0 as $R \to \infty$. Thus, from (24),

$$0 = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\mathcal{L}_R} H(s)\frac{x}{s(s+1)}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(s)\frac{x}{s(s+1)}ds$$

as required. ■